Student Number:
Teacher:

St George Girls High School

Mathematics Extension 2

2023 Trial HSC Examination

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in **Section I**, use the Multiple-Choice answer sheet provided

  For questions in **Section II:**
  - Answer the questions in the booklets provided
  - Start each question in a new writing booklet
  - Show relevant mathematical reasoning and/or calculations
  - Marks may not be awarded for incomplete or poorly presented solutions, or where multiple solutions are provided

Total marks: 100

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1– 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 –12)

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Q1-10	/10
Q11	/15
Q12	/16
Q13	/15
Q14	/15
Q15	/15
Q16	/14
TOTAL	**/100**
	%

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet provided for Questions 1 to 10.

1. What is the value of $\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)e^{i\frac{\pi}{3}}$?

(A) $e^{i\frac{11\pi}{12}}$

(B) $\frac{1}{\sqrt{2}}e^{i\frac{-11\pi}{12}}$

(C) $e^{i\frac{\pi}{4}}$

(D) $e^{i\frac{-11\pi}{12}}$

2. Each pair of lines given below intersects at $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

Which pair of lines are perpendicular?

(A) ℓ_1: $\underset{\sim}{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}-1\\-2\\1\end{pmatrix}$ and ℓ_2: $\underset{\sim}{r} = \begin{pmatrix}3\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\0\end{pmatrix}$

(B) ℓ_1: $\underset{\sim}{r} = \begin{pmatrix}2\\4\\6\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$ and ℓ_2: $\underset{\sim}{r} = \begin{pmatrix}-2\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\3\end{pmatrix}$

(C) ℓ_1: $\underset{\sim}{r} = \begin{pmatrix}0\\-3\\2\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\-3\end{pmatrix}$ and ℓ_2: $\underset{\sim}{r} = \begin{pmatrix}-1\\-2\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$

(D) ℓ_1: $\underset{\sim}{r} = \begin{pmatrix}3\\-2\\8\end{pmatrix} + \lambda\begin{pmatrix}-2\\4\\-5\end{pmatrix}$ and ℓ_2: $\underset{\sim}{r} = \begin{pmatrix}3\\4\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\1\end{pmatrix}$

3. Consider this statement.

"If I don't make my bed then my mum will take my iPhone away from me."

Which of the following is the converse of the contrapositive of the above statement?

(A) "If I don't make my bed then my mum will not take my iPhone away from me."

(B) "If my mum takes my iPhone away from me then I won't make my bed"

(C) "If I do make my bed then my mum will not take my iPhone away from me."

(D) "If I do make my bed then my mum will take my iPhone away from me."

4. Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ have components $\begin{pmatrix} 0.6 \\ 0 \\ t \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix}$ respectively.

The following two statements are made about $\boldsymbol{u}$ and $\boldsymbol{v}$:

(1) when $t = -1$, $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel.

(2) when $t = -0.8$, $\boldsymbol{u}$ is a unit vector.

Which of the following is true?

(A) Neither statement is correct.

(B) Only statement (1) is correct.

(C) Only statement (2) is correct.

(D) Both statements are correct.

5. Which expression is equal to $\int \frac{1}{\sqrt{-16x - 4x^2}}\, dx$?

(A) $\frac{1}{2} \sin^{-1}\left(\frac{x+2}{4}\right) + c$

(B) $\frac{1}{4} \sin^{-1}\left(\frac{x-2}{2}\right) + c$

(C) $\frac{1}{4} \sin^{-1}\left(\frac{x+2}{2}\right) + c$

(D) $\frac{1}{2} \sin^{-1}\left(\frac{x+2}{2}\right) + c$

6. Evaluate $(1+i)^{40}+(1-i)^{40}$.

(A) 2^{21}

(B) 2^{20}

(C) 2^{19}

(D) 2^{18}

7. Using a suitable trigonometric substitution, $\int_0^{\frac{\pi}{3}} \cot^3 x \sec^2 x \, dx$ can be expressed in which of the following ways?

(A) $\int_0^{\sqrt{3}} u^3 du$

(B) $\int_0^{\frac{\pi}{3}} \frac{1}{u^3} du$

(C) $\int_0^{\frac{1}{\sqrt{3}}} u^{-3} du$

(D) $\int_0^{\sqrt{3}} \frac{1}{u^3} du$

8. Which of the following statements is correct?

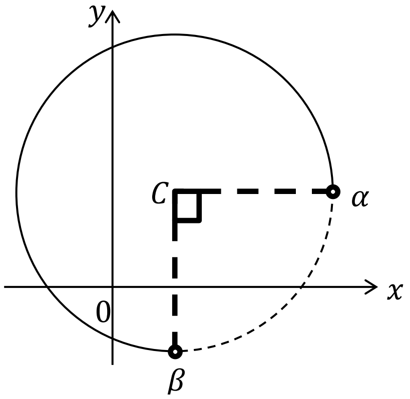
(A) $\forall a, b \in \mathbb{R}$ $\quad \sin a < \sin b \Rightarrow a < b$

(B) $\forall a, b \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\quad \sin a < \sin b \Rightarrow a < b$

(C) $\forall a, b \in \mathbb{R}$ $\quad \cos a < \cos b \Rightarrow a < b$

(D) $\forall a, b \in [0, \pi]$ $\quad \cos a < \cos b \Rightarrow a < b$

9. The diagram shows the solution of an equation, where C is the centre of the circle.



Which of these could be the equation?

(A) $\text{Arg}\,(z-\alpha)-\text{Arg}\,(z-\beta)=0$

(B) $\text{Arg}\,(z-\alpha)-\text{Arg}\,(z-\beta)=\frac{\pi}{2}$

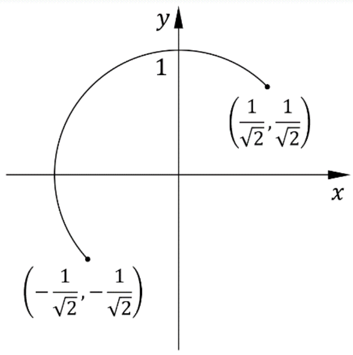
(C) $\text{Arg}\,(z-\alpha)-\text{Arg}\,(z-\beta)=\frac{\pi}{4}$

(D) $\text{Arg}\,(z-\beta)-\text{Arg}\,(z-\alpha)=\frac{\pi}{4}$

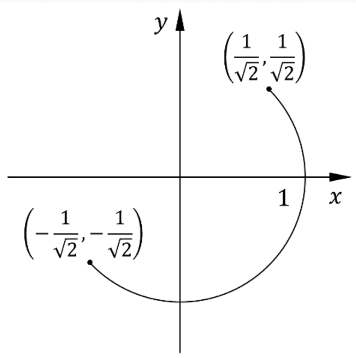
10. Which diagram best shows the curve described by the position vector

$$\underset{\sim}{r}(t)=\sin(t)\,\underset{\sim}{i}-\cos(t)\,\underset{\sim}{j}\text{ for }\frac{\pi}{4}\le t\le\frac{5\pi}{4}\,?$$

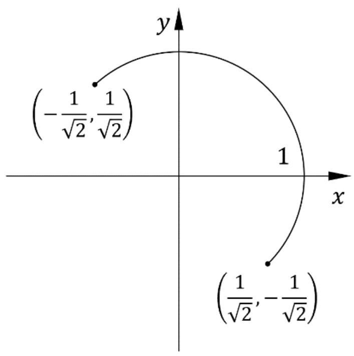
(A)



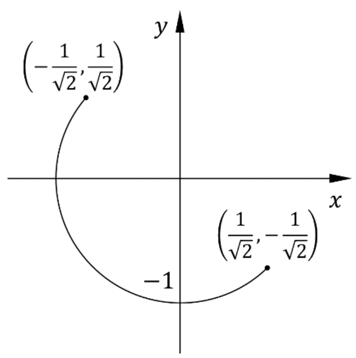
(B)



(C)



(D)



END OF SECTION I

Section II
90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the complex numbers $z_1 = 5 + i$ and $z_2 = -2 + i$.

Find the value of $\dfrac{\overline{z_1}}{z_1 + z_2}$, giving your answer in the form $a + ib$. 2

(b) For the complex number $z = 3e^{i\frac{\pi}{6}}$, write $\overline{z}^{\,4}$ in the form $a + ib$. 2

(c) Find the exact value of $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{3x}{\sqrt{x^2 + 2}}\,dx$. 3

(d) (i) Find the two square roots of $2i$, giving your answers in the form $x + iy$, where x and y are real numbers. 2

(ii) Hence, solve $2z^2 + 2\sqrt{2}z + 1 - i = 0$. Give your answers in the form $x + iy$. 2

(e) Consider the two vectors $\underset{\sim}{u} = 2\alpha\,\underset{\sim}{i} + (3 - \alpha)\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$,

where α is a scalar.

(i) Find $\underset{\sim}{p}$, the vector projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$. 2

(ii) Given that $3\left|\underset{\sim}{u}\right| = \sqrt{17}\left|\underset{\sim}{p}\right|$, find the value of α. 2

Question 12 (16 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use mathematical induction to prove that for all positive odd integers n, $n(n^2+1)$ is even. 3

(b) The polynomial $g(z) = z^4 + 2z^3 + 6z^2 + 8z + 8$ has roots $a + bi$ and $2ci$, where a, b and c are all real.

(i) Find all the roots of $g(z)$. 3

(ii) Write $g(z)$ as a product of two real quadratic factors. 2

(c) (i) Find the values of a, b, and c such that: 3

$$\frac{2x}{(x-4)(x+2)^2} = \frac{a}{x-4} + \frac{b}{x+2} + \frac{c}{(x+2)^2}.$$

(ii) Hence find $\int \frac{x}{(x-4)(x+2)^2}\,dx$. 2

(d) Shade the region on the Argand diagram where the inequalities $|z - 2 - 2i| \leq 2$ and $|\mathrm{Im}(z - 2i)| \geq 1$ hold simultaneously. 3

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Find all solutions to the equation $z^5 = -1$. 3

Give your answers in modulus-argument form.

(ii) Hence prove that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$. 2

(b) Use integration by parts to find $\int xe^{-2x}\ dx$. 3

(c) Prove by contradiction that if p is an integer, then $p^2 + 6$ is not divisible by 4. 3

(d) The line l has equation $\underset{\sim}{v} = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ b \\ 2 \end{pmatrix}$, where μ is a parameter and b is a constant.

The points P and Q have coordinates $(-1, 2, 3)$ and $(-2, 4, 2)$ respectively.

(i) Find a vector equation of the line PQ. 1
[Leave your answer in the form $\underset{\sim}{r} = a + \lambda b$]

(ii) Find the value of b for which the acute angle between line l and the line PQ is $\cos^{-1}\frac{1}{6}$. 3

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Calculate the modulus and argument of the sum of the roots of the equation $(4+3i)z^2 - (3-i)z - (4+2i) = 0$ in exact form. 3

(b) (i) If $t = \tan\frac{\theta}{2}$, show that $\frac{d\theta}{dt} = \frac{2}{1+t^2}$. 2

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{3}{8\cos\theta + 10}\, d\theta$. 3

Leave your answer correct to 3 significant figures.

(c) Consider the line $\boldsymbol{L}$ with position vector

$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

The sphere with equation $(x-3)^2 + (y+2)^2 + (z-4)^2 = 18$ intersects the line $\boldsymbol{L}$ at the points A and B.

Find the equation of the sphere with diameter AB. 4

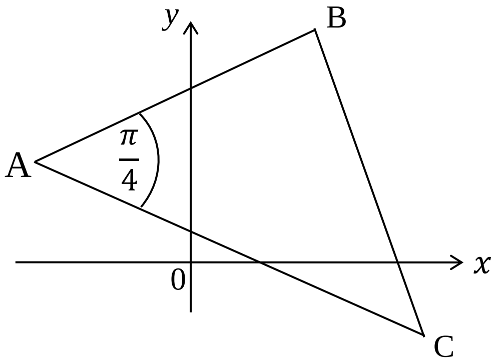
(d) Use mathematical induction to prove that $n! \geq 2^{n-1},\ n \in \mathbb{Z}^+$. 3

Question 15 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Prove by induction that $T_n = 3(2^n) + 1$ for $n \geq 1$, given $T_1 = 7$ and $T_n = 2T_{n-1} - 1$ for $n \geq 2$. 3

(b) The vertices A, B and C of triangle ABC are represented in the Argand diagram by the complex numbers a, b and c respectively. 3

AC = $\sqrt{2}$ AB and $\angle$CAB = $\frac{\pi}{4}$.



By using vectors, or otherwise, show that $c = b\,(1 - i) + a\,i$.

(c) Using the substitution $x = 2 + 2\cos^2\theta$, calculate the value of 4

$$\int_2^3 \sqrt{\frac{x-2}{4-x}}\,dx\,.$$

(d) (i) Let $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$ for $n = 0, 1, 2, 3, \ldots$

Show that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$ for every integer $n \geq 2$. 3

(ii) Hence, show that $I_4 = \left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24$. 2

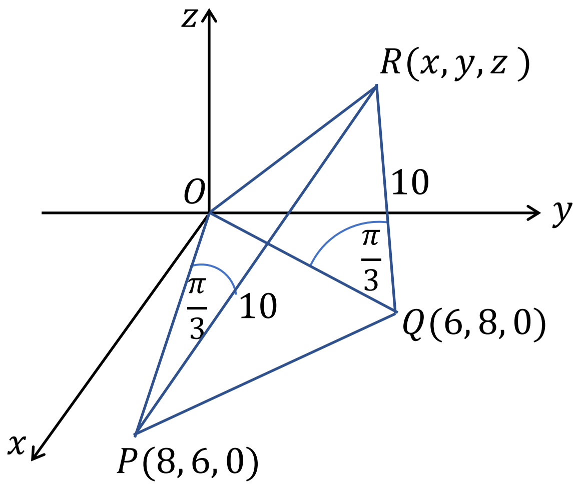
Question 16 (14 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. 2

(ii) By choosing a suitable trigonometric substitution, determine the value of 3

$$\int_0^1 \frac{dx}{x+\sqrt{1-x^2}}.$$

(b) The diagram shows a triangular pyramid with vertices $O(0,0,0)$, $P(8,6,0)$, $Q(6,8,0)$ and $R(x,y,z)$, where x, y and z are positive real numbers.



Given that $\angle RPO = \angle RQO = \frac{\pi}{3}$, $|\overrightarrow{PR}| = |\overrightarrow{QR}| = 10$ units, find the coordinates of R. 4

(c) (i) Using the inequality $\frac{x+y}{2} \geq \sqrt{xy}$, show that 3

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 \geq 3\left(\sqrt{ab}+\sqrt{ac}+\sqrt{bc}\right),$$

where a, b and c are positive numbers.

(ii) Hence, or otherwise, show that 2

$$(m^3p^3+m^3r^3+p^3r^3)^2 \geq m^3p^3r^3(m^3+r^3+p^3).$$

END OF EXAMINATION

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SUGGESTED SOLUTIONS

Section 1

Multiple Choice Answer Key

Question	Answer
1	D
2	A
3	C
4	D
5	D
6	A
7	D
8	B
9	C
10	C

Section I

1. $e^{\frac{3\pi}{4}i} \times e^{\frac{\pi}{3}i} = e^{\left(\frac{3\pi}{4}+\frac{\pi}{3}\right)i}$

$= e^{\frac{13\pi}{12}i}$

$= e^{-\frac{11\pi}{12}i}$

D

2. Looking for the dot product of the direction vectors being zero.

A: $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$= -2 + 2 + 0$

$= 0$ so A

A

3. The contrapositive of the statement is:

'If my mum does not take the iPhone away from me, then I will make my bed.'

Hence, the **converse** of the contrapositive is:

'If I do make my bed, then my mum will not take the iPhone away from me.'

C

4. $\underset{\sim}{u} = \begin{pmatrix} 0.6 \\ 0 \\ t \end{pmatrix}$ $\underset{\sim}{v} = \begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix}$

when $t = -1$ $\underset{\sim}{u} = \begin{pmatrix} 0.6 \\ 0 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 3/5 \\ 0 \\ -1 \end{pmatrix}$

Now $5 \times \begin{pmatrix} 3/5 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix}$ $\therefore$ (1) true

when $t = -0.8$, $\underset{\sim}{u} = \begin{pmatrix} 0.6 \\ 0 \\ -0.8 \end{pmatrix}$

$|\underset{\sim}{u}| = \sqrt{(0.6)^2 + (0.8)^2} = 1$ $\therefore$ unit vector $\therefore$ (2) true

$\therefore$ Both statements are true.

D

5. $\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}} dx = sin^{-1}\frac{f(x)}{a} + c$

$\int \frac{1}{\sqrt{-16x - 4x^2}} dx = \int \frac{1}{2\sqrt{-4x - x^2}} dx$

$= \int \frac{1}{2\sqrt{4 - 4 - 4x - x^2}} dx$

$= \int \frac{1}{2\sqrt{2^2 - (4 + 4x + x^2)}} dx$

$= \int \frac{1}{2\sqrt{2^2 - (x+2)^2}} dx$

$= \frac{1}{2} \sin^{-1}\left(\frac{x+2}{2}\right) + c$

D

6. $z = 1 + i.\ z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(cos\frac{\pi}{4} + i\,sin\frac{\pi}{4}\right)$

$\bar{z} = 1 - i = \sqrt{2}\left(cos\frac{\pi}{4} - i\,sin\frac{\pi}{4}\right).$

Using de Moivre's theorem $z^{40} = 2^{20}\text{cis}(10\pi)$,

$(\bar{z})^{40} = 2^{20}\text{cis}(-10\pi).$

$z^{40} + (\bar{z})^{40} = z^{40} + \overline{(z^{40})}$

$= 2\,Re(z^{40}) = 2^{21}\,cos(10\pi).$

$\therefore (1+i)^{40} + (1-i)^{40} = 2^{21}\,cos(10\pi).$

$= 2^{21}$

A

7. $\displaystyle\int_0^{\frac{\pi}{3}} \cot^3 x \sec^2 x \, dx$

$\displaystyle = \int_0^{\frac{\pi}{3}} \frac{1}{\tan^3 x} \sec^2 x \, dx$

$\displaystyle = \int_0^{\sqrt{3}} \frac{1}{u^3} \, du$

Let $u = \tan x$

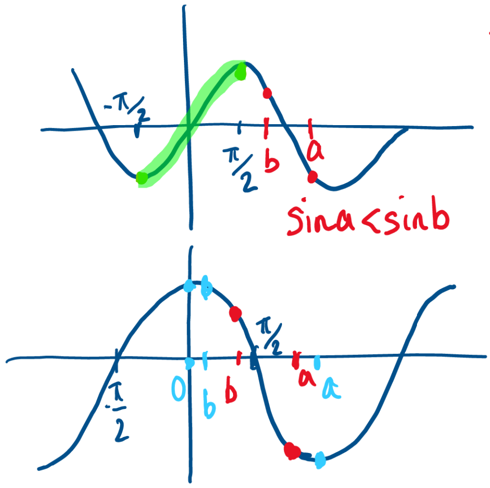
$du = \sec^2 x$

when $x = 0$, $u = \tan 0$, $u = 0$

$x = \frac{\pi}{3}$, $u = \tan\frac{\pi}{3} = \sqrt{3}$

D

8. The correct answer must be an increasing function in the whole domain, which only occurs for B. **B**



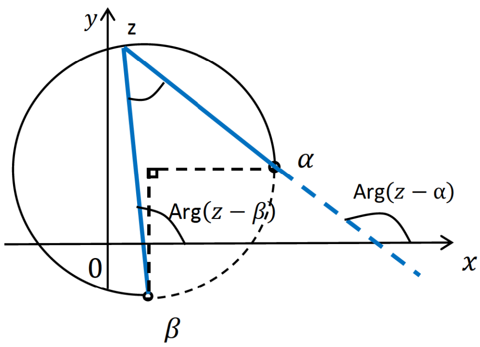
∴ A ✗

B ✓ Increasing function in the domain

C cos a < cos b but a > b ✗

D cos a < cos b but a > b ✗

9.



z lies on the circumference of a circle, radii from α and β meet at the centre at $\frac{\pi}{2}$ radians, this means the angle at z must be $\frac{\pi}{4}$ radians as the angle at circumference is half the angle at the when subtended by the same arc.

$z - \alpha$ represents the vector from α to z.

$z - \beta$ represents the vector from β to z.

Using the exterior angle of a triangle, we can see that the angle at $z = Arg(z - \alpha) - Arg(z - \beta)$.

C

10. $\underset{\sim}{r}\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} \times \underset{\sim}{i} - \cos\frac{\pi}{4} \times \underset{\sim}{j} = \frac{1}{\sqrt{2}}\underset{\sim}{i} - \frac{1}{\sqrt{2}}\underset{\sim}{j} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\underset{\sim}{r}\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} \times \underset{\sim}{i} - \cos\frac{5\pi}{4} \times \underset{\sim}{j} = -\frac{1}{\sqrt{2}}\underset{\sim}{i} - \left(-\frac{1}{\sqrt{2}}\right)\underset{\sim}{j} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Also passes through:

$\underset{\sim}{r}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} \times \underset{\sim}{i} - \cos\frac{\pi}{2} \times \underset{\sim}{j} = \underset{\sim}{i} = (1, 0)$, so C. **C**

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) $\bar{z} = 5 - i$		This part was very well done
$\frac{\bar{z}}{z_1 + z_2} = \frac{5 - i}{5 + i + -2 + i}$		
$= \frac{5-i}{3+2i}$		
$= \frac{5-i}{3+2i} \times \frac{3-2i}{3-2i}$	1	
$= \frac{15 - 10i - 3i - 2}{9+4}$		
$= \frac{13 - 13i}{13}$		
$= \frac{13(1-i)}{13}$		
$= 1 - i$	1	2
b) $z = 3e^{\frac{\pi}{6}i}$		
$= 3(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$		
$\bar{z} = 3(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$		
$\bar{z}^4 = [3(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})]^4$		
$= 3^4(\cos\frac{4\pi}{6} - i\sin\frac{4\pi}{6})$	1	
$= 81(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3})$	½	
$= 81(-\frac{1}{2} - i\frac{\sqrt{3}}{2})$	½	
$= -\frac{81}{2} - \frac{81\sqrt{3}}{2}i$		2

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) Let $I = \int_{\sqrt{2}}^{\sqrt{3}} \frac{3x}{\sqrt{x^2+2}}\,dx$		Overall this question was done well.
$= \frac{3}{2}\int_{\sqrt{2}}^{\sqrt{3}} \frac{2x}{\sqrt{x^2+2}}\,dx$		
Let $u = x^2$		
$\frac{du}{dx} = 2x$		
$du = 2x\,dx$		
when $x = \sqrt{3}$, $u = 3$		
$x = \sqrt{2}$, $u = 2$ ½		
$\therefore\ I = \frac{3}{2}\int_{2}^{3} \frac{du}{\sqrt{u+2}}$ ½	1	
$= \frac{3}{2}\int_{2}^{3} (u+2)^{-\frac{1}{2}}\,du$		
$= \frac{3}{2}\left[2(u+2)^{\frac{1}{2}}\right]_{2}^{3}$	1	
$= 3\left(\sqrt{3+2} - \sqrt{2+2}\right)$		
$= 3\left(\sqrt{5} - \sqrt{4}\right)$		
$= 3\sqrt{5} - 3\times 2$	1	3
$= 3\sqrt{5} - 6$		
Alternative method.		
$I = \frac{3}{2}\int_{\sqrt{2}}^{\sqrt{3}} \frac{2x}{\sqrt{x^2+2}}\,dx$		
$= \frac{3}{2}\int_{\sqrt{2}}^{\sqrt{3}} 2x(x^2+2)^{-\frac{1}{2}}\,dx$	1	
$= \frac{3}{2}\left[\frac{(x^2+2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_{\sqrt{2}}^{\sqrt{3}}$	1	
$= \frac{3}{2}\left[2\sqrt{x^2+2}\right]_{\sqrt{2}}^{\sqrt{3}}$		
$= 3\left[\sqrt{(\sqrt{3})^2+2} - \sqrt{(\sqrt{2})^2+2}\right]$		
$= 3\left(\sqrt{5} - \sqrt{4}\right)$	1	
$= 3\sqrt{5} - 6$		

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
d) i) Let $(a+ib)^2 = 2i$		This part was well done.
$a^2 - b^2 + 2abi = 2i$		
Equating parts		
$a^2 - b^2 = 0$ --- (1)		
$ab = 1$ --- (2)	1	
From (2) $b = \frac{1}{a}$ sub in (1)		
$a^2 - \left(\frac{1}{a}\right)^2 = 0$		
$a^4 - 1 = 0$		
$(a^2-1)(a^2+1) = 0$		
but $a^2 = 1$ since a is real		
$a = \pm 1$		
when $a = 1$, $b = 1$		
$a = -1$, $b = -1$		2
$\therefore \sqrt{2i} = \pm(1+i)$	1	
ii) $z = \dfrac{-2\sqrt{2} \pm \sqrt{(2\sqrt{2})^2 - 4\times 2\times(1-i)}}{2\times 2}$		
$= \dfrac{-2\sqrt{2} \pm \sqrt{8-8+8i}}{4}$		
$= \dfrac{-2\sqrt{2} \pm \sqrt{8i}}{4}$		
$= \dfrac{-2\sqrt{2} \pm 2\sqrt{2i}}{4}$	1	
$= \dfrac{-\sqrt{2} \pm \sqrt{2i}}{2}$		
$= \dfrac{-\sqrt{2} \pm (1+i)}{2}$ from (i)	½	
$= \dfrac{-\sqrt{2}+1+i}{2}$ or $\dfrac{-\sqrt{2}-1-i}{2}$	½	
$\therefore z = \dfrac{1-\sqrt{2}}{2} + \dfrac{i}{2}$ or $z = \dfrac{-\sqrt{2}-1}{2} - \dfrac{i}{2}$		2

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
e) i) $\underset{\sim}{u} = 2\alpha\underset{\sim}{i} + (3-\alpha)\underset{\sim}{k}$		
$\underset{\sim}{v} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$		
Now $\underset{\sim}{p} = \text{proj}_{\underset{\sim}{v}}\, \underset{\sim}{u}$		
$= \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\lvert\underset{\sim}{v}\rvert^2} \times \underset{\sim}{v}$		
$= \dfrac{2\alpha\times 2 + (3-\alpha)\times 1}{\left(\sqrt{2^2+2^2+1^2}\right)^2} \times \begin{pmatrix}2\\-2\\1\end{pmatrix}$	½ mk	for correct $\underset{\sim}{u}\cdot\underset{\sim}{v}$
$= \dfrac{4\alpha+3-\alpha}{4+4+1}\begin{pmatrix}2\\-2\\1\end{pmatrix}$	½ mk	for finding $\lvert\underset{\sim}{v}\rvert^2$ Many students did not square $\lvert\underset{\sim}{v}\rvert$
$= \dfrac{3\alpha+3}{9}\begin{pmatrix}2\\-2\\1\end{pmatrix}$		
$= \dfrac{3(\alpha+1)}{9}\begin{pmatrix}2\\-2\\1\end{pmatrix}$		
$= \dfrac{\alpha+1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}$	1 mk	2
ii) $\lvert\underset{\sim}{u}\rvert = \sqrt{(2\alpha)^2+(3-\alpha)^2}$		
$= \sqrt{4\alpha^2+9-6\alpha+\alpha^2}$		
$= \sqrt{5\alpha^2-6\alpha+9}$		
and $\lvert\underset{\sim}{p}\rvert = \dfrac{\alpha+1}{3} \times \sqrt{2^2+(-2)^2+1}$	½ mk	Many students did not multiply $\sqrt{2^2+(2)^2+1}$ by $\dfrac{\alpha+1}{3}$ and therefore have made the question easier.
$= \dfrac{\alpha+1}{3} \times \sqrt{4+4+1}$		
$= \dfrac{\alpha+1}{3} \times \sqrt{9}$		
$= \dfrac{\alpha+1}{3} \times 3\sqrt{17}$ $\quad \lvert\underset{\sim}{p}\rvert$		
$= \alpha+1$ $\quad (\alpha+1)$		
Now $3\lvert\underset{\sim}{u}\rvert =$		
$3\sqrt{5\alpha^2-6\alpha+9} = \sqrt{17}$		

MATHEMATICS EXTENSION 2 – QUESTION 11		
SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
Squaring both sides		
$9(5\alpha^2-6\alpha+9) = 17(\alpha+1)^2$		
$45\alpha^2-54\alpha+81 = 17(\alpha^2+2\alpha+1)$		½ mark
$= 17\alpha^2+34\alpha+17$		
$28\alpha^2-88\alpha+64=0$		
$7\alpha^2-22\alpha+16=0$		
$(7\alpha-8)(\alpha-2)=0$		1 mark
$\alpha = 2$ or $\dfrac{8}{7}$		
		2

MATHEMATICS EXTENSION 2 – QUESTION 12

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
Step 1 – Prove true for $n=1$		
$x(x^2+1) \quad x \to n \quad n(n^2+1)$		
$n=1$		
$= 1(1^2+1)$		
$= 2$ which is even.	½	Correctly proves the base case
Step 2 – Assume true for $n=k$ where k is odd		
ie Assume $k(k^2+1) = 2M$ where $M \in \mathbb{Z}$	½	Correctly states the assumption
Step 3 – Prove true for $n=k+2$		
ie Prove $(k+2)[(k+2)^2+1]$ is even $k \in$ odd integer		
$(k+2)[(k+2)^2+1]$	½	Correctly states what is being proved.
$= (k+2)(k^2+4k+4+1)$		
$= (k+2)(k^2+4k+5)$		
$= k^3+4k^2+5k+2k^2+8k+10$		
$= k^3+6k^2+13k+10$		
$= k^3+k + 6k^2+12k+10$		
$= k(k^2+1) + 2(3k^2+6k+5)$		
$= 2M + 2(3k^2+6k+5)$ by the assumption	1	Correctly uses the assumption in the proof
$= 2(M+3k^2+6k+5)$ $M \in \mathbb{Z}$, k is a positive odd integer		
which is even	½	Correctly completes proof and includes conclusion.
$\therefore$ true for $n=k+2$, if true for $n=k$		
Step 4		
$\therefore$ By the Principal of Mathematical Induction		
it is true for all positive odd integers n.		

MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
OR		
Step 3 Prove true for $n = 2k+1$		
i.e. Prove $(2k+1)[(2k+1)^2+1]$ is even		
$(2k+1)[(2k+1)^2+1]$		
$= (2k+1)(4k^2+4k+1+1)$		
$= 8k^3+8k^2+4k+4k^2+4k+2$		
$= 8k^3+12k^2+8k+2$ $k(k^2+1) = 2M$		
$= 8(2M-k)+12k^2+8k+2$ $k^3+k = 2M$, $k^3 = 2M-k$		
$= 16M-8k+12k^2+8k+2$		
$= 16M+12k^2+2$		
$= 2(8M+6k^2+1)$ which is even $M \in \mathbb{Z}$, k is an odd positive integer		
etc . . .		

MATHEMATICS EXTENSION 2 – QUESTION 12

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b)(i) $g(z)=z^4+2z^3+6z^2+8z+8$ $\quad a,b,c\in\mathbb{R}$		
If $a+bi$ is a root then $a-bi$ is a root.		
If $2ci$ is a root then $-2ci$ is a root.		
This is because the roots occur in conjugate pairs when the coefficients		
are real.		
$\alpha+\beta+\gamma+\delta=\frac{-b}{a}$		
$a+ib+a-ib+2ci-2ci=-2$		to get all conjugate pairs
$2a=-2$		and correctly evaluates
$a=-1$	1	$a=-1$
$\alpha\beta\gamma\delta=\frac{e}{a}$		
$(a+ib)(a-ib)(2ci)(-2ci)=8$	½	correct expression for
$4c^2(a^2+b^2)=8 \quad \ldots$ (1)		product of roots
$\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\alpha\gamma\delta$	½	correct expression for
$(a+ib)(a-ib)(2ci)+(a+ib)(a-ib)(-2ci)+(a-ib)(2ci)(-2ci)+(a+ib)(2ci)(-2ci)=-8$		either sum of the
$\cancel{2ci(a^2+b^2)}-\cancel{2ci(a^2+b^2)}+4c^2(a-ib)+4c^2(a+ib)=-8$		roots two at a time or
$4ac^2-\cancel{4b}c^2i+4ac^2+\cancel{4b}c^2i=-8$		three at a time.
$8ac^2=-8$		
$ac^2=-1$		
$a=-1 \quad \therefore (-1)c^2=-1$		
$c^2=1$		
$c=\pm1$	½	Correctly evaluates
$\therefore$ From (1) $4c^2(a^2+b^2)=8$		$c=\pm1$
$4c^2((-1)^2+b^2)=8$		
$c^2(1+b^2)=2$		
For $c=\pm1$ $\quad (\pm1)^2(1+b^2)=2$		
$1+b^2=2$		
$b^2=1$		Correctly evaluates
$b=\pm1$	½	$b=\pm1$
Now roots are $a\pm bi$, $\pm2ci$		
$-1\pm i$, $\pm2i$ $\quad$ (ie $-1+i$, $-1-i$, $2i$, $-2i$)		* ½ mark lost if roots are not written

MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
$\alpha \quad \beta \quad \gamma \quad \delta$		
OR		
$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=6$		
$(a+ib)(a-ib)+(a+ib)(-2ci)+(2ci)(a+ib)+(a-ib)(2ci)+(a-ib)(-2ci)+(2ci)(-2ci)=6$		
$a^2+b^2-\cancel{2aci}+\cancel{2bc}+\cancel{2aci}-\cancel{2bc}+\cancel{2aci}+\cancel{2bc}-\cancel{2aci}-\cancel{2bc}+4c^2=6$		
$a^2+b^2+4c^2=6$		
$(-1)^2+b^2+4(\pm1)^2=6$		
$1+b^2+4=6$		
$b^2=1$ Note if you got $b=\pm\sqrt{3}$, $c=\pm\sqrt{2}$		
$b=\pm1$ These solutions do not give zeros when substituted in.		
(ii) $g(z)=[z-(-1+i)][z-(-1-i)](z-2i)(z+2i)$		
$=(z+1-i)(z+1+i)(z-2i)(z+2i)$		
$=[(z+1)-i][(z+1)+i](z^2+4)$		
$=[(z+1)^2+1](z^2+4)$		
$=(z^2+2z+2)(z^2+4)$		
NOTE:		
$a^2+b^2+4c^2=6$		
$1+3+4c^2=6$		
$4c^2=2$		
$c^2=\frac{1}{2}$		
$c=\pm\frac{1}{\sqrt{2}}$		
$P(\sqrt{2}i)=(\sqrt{2}i)^4+2(\sqrt{2}i)^3+6(\sqrt{2}i)^2+8(\sqrt{2}i)+8$		
$=4-4\sqrt{2}i-12+8\sqrt{2}i+8\neq0$		
$\neq0.$		
$P(2i)=(2i)^4+2(2i)^3+6(2i)^2+8(2i)+8$		
$=16-\cancel{16i}-24+\cancel{16i}+8$		
$=0$		

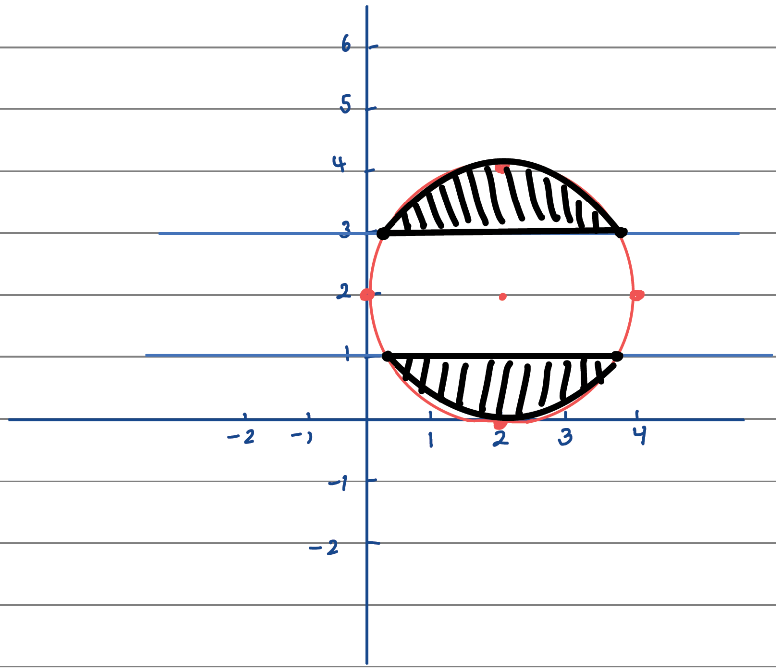
MATHEMATICS EXTENSION 2 – QUESTION 12

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) (i) $\dfrac{2x}{(x-4)(x+2)^2} = \dfrac{a}{x-4} + \dfrac{b}{x+2} + \dfrac{c}{(x+2)^2}$		
$2x = a(x+2)^2 + b(x-4)(x+2) + c(x-4)$	1 mark	correct equation
Sub in $x=-2$		
$-4 = -6c$		
$c = \frac{2}{3}$	1 mark	to find one of the values
Sub in $x = 4$		
$8 = 36a$		
$a = \frac{2}{9}$		
Sub in $x = 0$		
$0 = 4a - 8b - 4c$		
$0 = 4(2/9) - 8b - 4(2/3)$		
$0 = 8 - 72b - 24$		
$72b = -16$		
$b = \frac{-2}{9}$		
$\therefore\ a = \frac{2}{9},\ b = \frac{-2}{9},\ c = \frac{2}{3}$	1 mark	find all correct values
(ii) $\displaystyle\int \frac{x}{(x-4)(x+2)^2}\,dx = \frac{1}{2}\int \frac{2x}{(x-4)(x+2)^2}\,dx$	1 mark	to establish correct integral to enable use of (i)
$\displaystyle = \frac{1}{2}\int \frac{2}{9(x-4)}\,dx - \frac{1}{2}\int \frac{2}{9(x+2)}\,dx + \frac{1}{2}\int \frac{2}{3(x+2)^2}\,dx$		
$\displaystyle = \frac{1}{9}\int \frac{1}{x-4}\,dx - \frac{1}{9}\int \frac{1}{x+2}\,dx + \frac{1}{3}\int (x+2)^{-2}\,dx$		
$\displaystyle = \frac{1}{9}\ln\lvert x-4\rvert - \frac{1}{9}\ln\lvert x+2\rvert + \frac{1}{3}\frac{(x+2)^{-1}}{-1} + C$		
$\displaystyle = \frac{1}{9}\ln\left\lvert\frac{x-4}{x+2}\right\rvert - \frac{1}{3(x+2)} + C$	1	Correct answer. NOTE: Some half marks awarded for a minor error within working.

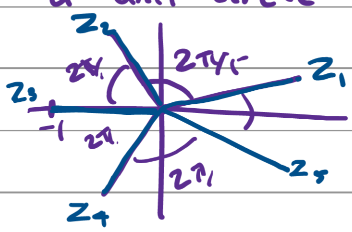
MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
(i) METHOD 2		
$\frac{2x}{(x-4)(x+2)^2} = \frac{a}{x-4} + \frac{b}{x+2} + \frac{c}{(x+2)^2}$		
$2x = a(x+2)^2 + b(x-4)(x+2) + c(x-4)$		
$2x = ax^2 + 4ax + 4a + bx^2 - 2bx - 8b + cx - 4c$		
$0x^2 + 2x + 0 = (a+b)x^2 + (4a-2b+c)x + 4a - 8b - 4c$		
By equating coefficients		
$a+b=0$ $4a-2b+c=2$ $4a-8b-4c=0$		
$b=-a$ $4a+2a+c=2$ $a-2b-c=0$		
$6a+c=2$ $a+2a-c=0$		
sub $3a=c$		
$\therefore 6a+3a=2$		
$9a=2$		
$\therefore a=2/9,\ b=-2/9,\ c=\frac{2}{3}$		

MATHEMATICS EXTENSION 2 – QUESTION 12

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
d) $\|z-(2+2i)\| \le 2$ Circle centre $(2,2)$, radius $=2$		
	1 mark 1 mark 1 mark	correct circle one correct line + shading Correct graph
$\|Im(z-2i)\| \ge 1$ $\|x+iy-2i\| \ge 1$ $\|x+(y-2)i\| \ge 1$ $\|y-2\| \ge 1$ $y-2 \ge 1$ or $-(y-2) \ge 1$ $y \ge 3$ $y-2 \le -1$ $y \le 1$		

MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a)		
Let $z = r(\cos\theta + i\sin\theta)$ for $-\pi < \theta \le \pi$		
Now $z^5 = r^5(\cos\theta + i\sin\theta)^5$		
$= r^5 \operatorname{cis} 5\theta$		
but $z^5 = -1$		
$z^5 = 1(\operatorname{cis}\pi)$		
$\therefore\ r^5 \operatorname{cis} 5\theta = \operatorname{cis}\pi$		
$r^5 = 1$, $\quad 5\theta = \pi + 2k,\ k \in \mathbb{Z}$		
$\theta = \frac{\pi}{5} + \frac{2\pi}{5}k$		
$= \frac{\pi}{5}(1+2k)$		
$-\pi < \frac{\pi}{5}(1+2k) \le \pi$		
$-5 < 1+2k \le 5$		
$-6 < 2k \le 4$		
$-3 < k \le 2$		
$z = \operatorname{cis} \frac{\pi(1+2k)}{5}$,	1	
when $k=0$, $z = \operatorname{cis}\frac{\pi}{5}$		
$k=1$, $z = \operatorname{cis}\frac{3\pi}{5}$		
$k=2$, $z = \operatorname{cis}\frac{5\pi}{5} = \operatorname{cis}\pi$		• Many students did not leave the solution as a principle argument
$k=-1$, $z = \operatorname{cis}\left(-\frac{\pi}{5}\right)$		• Marks were still awarded but
$k=-2$, $z = \operatorname{cis}\left(-\frac{3\pi}{5}\right)$		remember to leave roots in
$\therefore$ solutions are:		
$z = \operatorname{cis}-\frac{3\pi}{5},\ \operatorname{cis}-\frac{\pi}{5},\ -1,\ \operatorname{cis}\frac{\pi}{5},\ \operatorname{cis}\frac{3\pi}{5}$	2	that form
Alternative Method	3mk - Correct solutions	
Solutions are evenly spaced around	2mk - 3 correct solutions incl. -1	
a unit circle by $\frac{2\pi}{5}$, starting at $\frac{\pi}{5}$	or -4 " " not incl. -1	
	1 mk - -1 as a solution	

$\therefore z = \operatorname{cis}\frac{\pi}{5},\ \operatorname{cis}\frac{3\pi}{5},\ \operatorname{cis}\pi,\ \operatorname{cis}\frac{7\pi}{5},\ \operatorname{cis}\frac{9\pi}{5}$

but $\operatorname{cis}\frac{7\pi}{5} = \operatorname{cis} -\frac{3\pi}{5}$ and $\operatorname{cis}\frac{9\pi}{5} = \operatorname{cis}-\frac{\pi}{5}$

$\therefore z = \operatorname{cis}-\frac{3\pi}{5},\ \operatorname{cis}-\frac{\pi}{5},\ \operatorname{cis}\pi,\ \operatorname{cis}\frac{\pi}{5},\ \operatorname{cis}\frac{3\pi}{5}$

MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) ii)		
Sum of the roots $= -\frac{b}{a}$		
ie		
$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5} + \cos-\frac{\pi}{5} + i\sin-\frac{\pi}{5} + \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} + \cos-\frac{3\pi}{5} + i\sin-\frac{3\pi}{5} = 0$	1 mk	- To show the sum of all the roots and that they equal to 0.
$\cos\frac{\pi}{5} + \cancel{i\sin\frac{\pi}{5}} + \cos\frac{\pi}{5} - \cancel{i\sin\frac{\pi}{5}} + \cos\frac{3\pi}{5} + \cancel{i\sin\frac{3\pi}{5}} + \cos\frac{3\pi}{5} - \cancel{i\sin\frac{3\pi}{5}} - 1 = 0$	½ mk	- A few students did not include this step
$2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} = 1$ $2(\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}) = 1$ $\therefore \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$	½ mk	- if previous steps are included and correct.
NB: As it is a 'show' question, students should include all steps in their working, particularly terms that are cancelled out.		
b) Using integration by parts: $\int uv'\,dx = uv - \int vu'\,dx$		This part was very well done.
For $\int xe^{-2x}\,dx$ Let $u = x$, $v' = e^{-2x}$ $u' = 1$, $v = -\frac{1}{2}e^{-2x}$	1	
$\int xe^{-2x}\,dx = x \times \left(-\frac{1}{2}e^{-2x}\right) - \int -\frac{1}{2}e^{-2x} \times 1\,dx$ $= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$	1	
$= -\frac{1}{2}xe^{-2x} + \frac{1}{2} \times -\frac{1}{2}e^{-2x} + C$		
$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$ $= \frac{-e^{-2x}}{2}\left(x + \frac{1}{2}\right) + C$	1	3

MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) We assume:		
If p is an integer, then p^2+6 is divisible by 4.	$\frac{1}{2}$	— mark for the assumption / negation)
So let $p^2+6=4M$ (✱) where M is an integer	$\frac{1}{2}$	
Case 1: If p is even then $p=2k$ where k is an integer		
From (✱) $LHS=(2k)^2+6$ $=4k^2+6$ $=4k^2+4+2$ $=4(k^2+1)+2$ which is not divisible by 4 $\therefore$ a contradiction from the assumption that p^2+6 is divisible by 4.	1	
Case 2: If p is odd the $p=2k+1$		
From (✱) $LHS=(2k+1)^2+6$ $=4k^2+4k+1+6$ $=4k^2+4k+7$ $=4k^2+4k+4+3$ $=4(k^2+k+1)+3$ which is not divisible by 4 $\therefore$ a contradiction from the assumption that p^2+6 is divisible by 4.	1	
$\therefore$ If p is an integer, then p^2+6 is not divisible by 4..		

MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) Alternative solution 1:		
From (*) $p^2+6 = 4M$		
$p^2 = 4M-6$		
$p^2 = 2(2M-3)$	$\frac{1}{2}$	
$\therefore$ p^2 is divisible by 2		
$\therefore$ p is divisible by 2	$\frac{1}{2}$	
Let $p = 2k$ where $k \in \mathbb{Z}$		
From (*) LHS $= (2k)^2 + 6$		
$= 4k^2 + 6$		
$= 4k^2 + 4 + 2$	1	
$= 4(k^2+1) + 2$		
which is not divisible by 4		
but $p^2+6 = 4M$		
$\therefore$ a contradiction from the assumption.	$\frac{1}{2}$	3
$\therefore$ If p is an integer, p^2+6 is not divisible by 4.		
NB: Many students got to		
$p^2 = 2(2M-3)$	$\frac{1}{2}$	
then took the square root to make p		No marks were
the subject, ie. $p = \pm\sqrt{2(2M-3)}$		given for any
then aimed to prove that		subsequent working
p was not an integer (or irrational)		after this line.
and ended the proof there saying		
it was a contradiction.		
This proof is not correct because		
the negation of the original		
statement is that: if p is an integer		
then p^2+6 is divisible by 4.		
It is already given that p is an integer.		

MATHEMATICS EXTENSION 2 – QUESTION 13		
SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) Alternative Solution 2		
To prove the statement is true using contradiction, we assume that $p^2 + 6$ is divisible by 4 and look for a contradiction. So let $p^2 + 6 = 4k, where\ k\ is\ an\ integer.$ We are to consider two different cases p odd or p even. **Case 1:** $if\ p$ is even then $p = 2r$ $, where\ r\ is\ an\ integer.$ This means $p^2 + 6 = 4k$ becomes $(2r)^2 + 6 = 4k$ $4r^2 + 6 = 4k$ t $2r^2 + 3 = 2k$ But $2r^2 + 3$ is odd, and $2k$ is even, so there is a contradiction here. **Case 2:** $if p$ is odd then $p = 2s + 1, where\ s\ is\ an\ integer.$ This means $p^2 + 6$ becomes $(2s + 1)^2 + 6 = 4k$ $4s^2 + 4s + 1 + 6 = 4k$ $2(2s^2 + 2s) + 7 = 4k$ But $2(2s^2 + 2s) + 7$ is odd, and $4k$ is even, so there is also a contradiction here.		

MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
d) i) $P(-1,2,3)$ $\quad Q(-2,4,2)$		
$\therefore \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$		
$= \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$		
$= \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$	$\frac{1}{2}$ mark	
$\therefore \overrightarrow{PQ}$ is the direction vector		
$\therefore \underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$	$\frac{1}{2}$ mark	
or $\underset{\sim}{r} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$	or	1
ii) The angle between the direction vectors is $\cos^{-1}\frac{1}{6}$. So if the direction vectors are:		
$\underset{\sim}{d_1} = \begin{pmatrix} -1 \\ b \\ 2 \end{pmatrix}$ and $\underset{\sim}{d_2} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$,		
then		
$\cos\theta = \dfrac{\begin{pmatrix} -1 \\ b \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}}{\left\|\begin{pmatrix} -1 \\ b \\ 2 \end{pmatrix}\right\| \cdot \left\|\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}\right\|}$	2	• Writing that if $\cos^{-1}\frac{1}{6} = \theta$ then $\frac{1}{6} = \cos\theta$ • Finding the dot product • Find the magnitude of $\underset{\sim}{d_1}$ and $\underset{\sim}{d_2}$
$\frac{1}{6} = \dfrac{1 + 2b - 2}{6}$		
$1 = 1 + 2b - 2$		
$2b = 2$	1	For answer 3
$b = 1$		

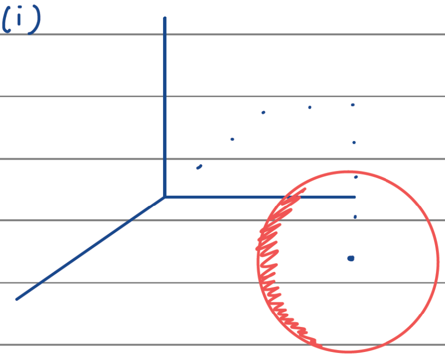
MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) $\alpha + \beta = \frac{-b}{a}$		
$= -\frac{-(3-i)}{4+3i}$		
$= \frac{3-i}{4+3i} \times \frac{4-3i}{4-3i}$		
$= \frac{12-9i-4i-3}{16+9}$		
$= \frac{9-13i}{25}$		
$= \frac{9}{25} - \frac{13}{25}i$ quad 4		
$\lvert\alpha+\beta\rvert = \sqrt{\left(\frac{9}{25}\right)^2 + \left(\frac{13}{25}\right)^2}$		
$= \sqrt{\frac{250}{225}}$		
$= \frac{5\sqrt{10}}{25}$		
$= \frac{\sqrt{10}}{5}$ or $\sqrt{\frac{2}{5}}$		
$\arg(\alpha+\beta) = -\tan^{-1}\left(\frac{13/25}{9/25}\right)$ quad 4		
$= -\tan^{-1}\left(\frac{13}{9}\right)$		

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b) (i) $t = \tan\frac{\theta}{2}$		
$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$ $\sin^2\theta + \cos^2\theta = 1$ $\therefore \tan^2\theta + 1 = \sec^2\theta$	1	correct derivative
$= \frac{1}{2}\left(\tan^2\frac{\theta}{2} + 1\right)$		
$= \frac{1}{2}(t^2+1)$		
$\frac{dt}{d\theta} = \frac{t^2+1}{2}$	1	Correctly showing $\frac{d\theta}{dt} = \frac{2}{1+t^2}$
$\therefore \frac{d\theta}{dt} = \frac{2}{1+t^2}$		
(ii) $\int_0^{\frac{\pi}{2}} \frac{3}{8\cos\theta + 10}\,d\theta$ $t = \tan\frac{\theta}{2}$, $\frac{d\theta}{dt} = \frac{2}{1+t^2}$ $\theta = \frac{\pi}{2} \to t = \tan\frac{\pi}{4}$, $t = 1$; $\theta = 0 \to t = \tan 0$, $t = 0$	1 mark	t-values
$= \int_0^1 \left(\frac{3}{8\left(\frac{1-t^2}{1+t^2}\right) + 10} \times \frac{2}{1+t^2}\right) dt$ $\therefore d\theta = \frac{2}{1+t^2}\,dt$	1 mark	correct integral in terms of t.
$= 3\int_0^1 \frac{1}{4(1-t^2) + 5(1+t^2)}\,dt$		
$= 3\int_0^1 \frac{1}{9+t^2}\,dt$		
$= 3\left[\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)\right]_0^1$		
$= \left[\tan^{-1}\left(\frac{t}{3}\right)\right]_0^1$		
$= \tan^{-1}\frac{1}{3} - \tan^{-1} 0$		
$= \tan^{-1}\frac{1}{3}$	1 mark	Correct answer.
$\doteqdot 0.322$		($\frac{1}{2}$ mark if not evaluated)

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) (i)  $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ $\underset{\sim}{r} = \begin{bmatrix} 3+\lambda \\ -2-\lambda \\ 1-\lambda \end{bmatrix}$		
Where does the sphere intersect L $\therefore x = 3+\lambda$, $y = -2-\lambda$, $z = 1-\lambda$ SUB IN $(x-3)^2 + (y+2)^2 + (z-4)^2 = 18$		
$(3+\lambda-3)^2 + (-2-\lambda+2)^2 + (1-\lambda-4)^2 = 18$ $\lambda^2 + \lambda^2 + (-3-\lambda)^2 = 18$ $2\lambda^2 + 9 + 6\lambda + \lambda^2 = 18$ $3\lambda^2 + 6\lambda - 9 = 0$ $\lambda^2 + 2\lambda - 3 = 0$ $(\lambda+3)(\lambda-1) = 0$ $\therefore \lambda = -3$ or $\lambda = 1$	1	Correctly using equations to solve for λ.
Let A be the point for when $\lambda = -3$ $\begin{pmatrix} 3-3 \\ -2+3 \\ 1+3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$		
Let B be the point for when $\lambda = 1$ $\begin{pmatrix} 3+1 \\ -2-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}$	1	Correctly finding A and B.
So the line intersects the sphere at		
$A(0,1,4)$ and $B(4,-3,0)$ Centre $\left(\frac{0+4}{2}, \frac{1-3}{2}, \frac{4+0}{2}\right)$ $C(2,-1,2)$	1	Correct centre
$d_{AB} = \sqrt{(4-0)^2 + (-3-1)^2 + (0-4)^2}$ $= \sqrt{16+16+16}$		

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
$d_{AB} = 4\sqrt{3} \quad \therefore \quad r = 2\sqrt{3}$		
$\therefore$ Equation of the sphere with diameter AB is		
$(x-2)^2 + (y+1)^2 + (z-2)^2 = (2\sqrt{3})^2$	1	Correct radius and equation
$(x-2)^2 + (y+1)^2 + (z-2)^2 = 12$		

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
d) Step 1 – Base case – Prove true for $n=1$		
$LHS = n!$ $\quad$ $RHS = 2^{n-1}$		
$= 1!$ $\quad$ $= 2^{1-1}$		
$= 1$ $\quad$ $= 2^0$		
$= 1$	½	Correctly showing the base case
$\therefore LHS \geq RHS$		
$\therefore$ True for $n=1$		
Step 2 – Inductive hypothesis – Assume true for $n=k$		Correctly states the assumption
ie $k! \geq 2^{k-1}$, $k \in \mathbb{Z}^+$	½	
Step 3 – Inductive step – Prove true for $n=k+1$		
ie Prove $k!(k+1) \geq 2^{k+1-1}$		
$(k+1)! \geq 2^k$	½	Correctly stating what is to be proved.
Consider $(k+1)! - 2^k$ $\quad$ Note $10-4=6$		
$= k!(k+1) - 2^k$ $\quad$ $\geq 8-4=4$		
$\geq 2^{k-1}(k+1) - 2^k$ By the assumption.	½	Correctly uses the assumption in the proof.
$= 2^{k-1}(k+1-2)$		
$= 2^{k-1}(k-1)$		
≥ 0 since $k \in \mathbb{Z}^+$ and $2^{k-1} > 0$ for all k	1	Correct setting out of proof with conclusion
$\therefore (k+1)! - 2^k \geq 0$		
$(k+1)! \geq 2^k$		
$\therefore$ true for $n=k+1$ if true for $n=k$		
Step 4 $\therefore$ By the principle of mathematical		
induction it is true for all $n \in \mathbb{Z}^+$		

MATHEMATICS EXTENSION 2 – QUESTION 14

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS

OR Step 3

Prove $(k+1)! \geq 2^k$

$k! \geq 2^{k-1}$

$10 \geq 8$

$10 \times 2 \geq 8$

$$
\begin{aligned}
LHS &= (k+1)! \\
&= k!(k+1) \\
&\geq 2^{k-1}(k+1) \\
&= k2^{k-1} + 2^{k-1} \\
&\geq 2^{k-1} + 2^{k-1} \\
&= 2 \cdot 2^{k-1} \\
&= 2^k
\end{aligned}
$$

$\therefore (k+1)! \geq 2^k$

OR

$\geq 2^{k-1}(1+1)$ since $k \in \mathbb{Z}^+ \therefore k \geq 1$

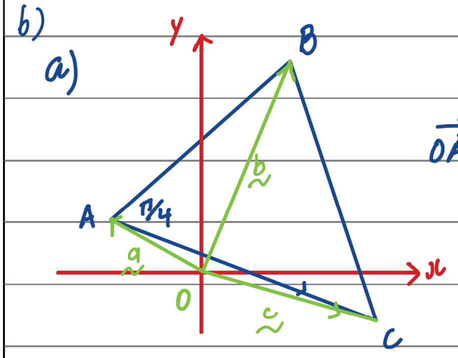
$= 2^{k-1} \cdot 2$

$= 2^k$

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) Step 1 – Base case – Prove true for $n=1$		
$T_n = 3(2^n) + 1$		
$n=1$ $\quad T_1 = 3(2^1) + 1$		
$= 7$		Correctly proves
$\therefore$ True for $n=1$	1	base case.
Step 2 – Assume true for $n=k$		
ie $\quad T_k = 3(2^k) + 1$	½	Correct assumption
Step 3 – Prove true for $n=k+1$		
ie Prove $T_{k+1} = 3(2^{k+1}) + 1$ given		
$T_{k+1} = 2T_k - 1$		
$LHS = T_{k+1}$	1	Correct set out
$= 2T_k - 1$		of proof
$= 2[3(2^k) + 1] - 1$		
$= 2 \times 3(2^k) + 2 - 1$	½	Correct use of
$= 3(2^{k+1}) + 1$		assumption within
$= RHS$		proof.
$\therefore$ True for $n=k+1$ if true for $n=k$		
Step 4		
$\therefore$ By mathematical induction it is true for		
all positive integer n.		

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b)		
a) 		
$\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$ $\quad$ $\lvert\overrightarrow{AC}\rvert = \sqrt{2}\,\lvert\overrightarrow{AB}\rvert$		
$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$		
$\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$	1	Some correct progress
To get $\overrightarrow{AB}$ we rotate vector $\overrightarrow{AC}$ anticlockwise by $\frac{\pi}{4}$		
and reduce it by a factor of $\frac{1}{\sqrt{2}}$ since $AC = \sqrt{2}AB$, $AB = \frac{1}{\sqrt{2}}AC$		
$\therefore \quad \overrightarrow{AB} = \frac{1}{\sqrt{2}}\overrightarrow{AC}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$	1	Correct rotation and reduction
$\underset{\sim}{b} - \underset{\sim}{a} = \frac{1}{\sqrt{2}}(\underset{\sim}{c} - \underset{\sim}{a})\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$		
$\underset{\sim}{b} - \underset{\sim}{a} = (\underset{\sim}{c} - \underset{\sim}{a})\left(\frac{1}{2} + \frac{1}{2}i\right)$		
$2(\underset{\sim}{b} - \underset{\sim}{a}) = (\underset{\sim}{c} - \underset{\sim}{a})(1+i)$		
$2\underset{\sim}{b} - 2\underset{\sim}{a} = \underset{\sim}{c} + \underset{\sim}{c}i - \underset{\sim}{a} - \underset{\sim}{a}i$	1	Correct process to show result.
$2\underset{\sim}{b} - \underset{\sim}{a} + \underset{\sim}{a}i = \underset{\sim}{c}(1+i)$		
$\underset{\sim}{c} = \dfrac{2\underset{\sim}{b} - \underset{\sim}{a} + \underset{\sim}{a}i}{1+i} \quad \dfrac{\times(1-i)}{\times(1-i)}$		
$\underset{\sim}{c} = \dfrac{(2\underset{\sim}{b} - \underset{\sim}{a})(1-i) + \underset{\sim}{a}i(1-i)}{1-i^2}$		
$\underset{\sim}{c} = \dfrac{2\underset{\sim}{b} - 2\underset{\sim}{b}i \cancel{- \underset{\sim}{a}} + \underset{\sim}{a}i + \underset{\sim}{a}i \cancel{+ \underset{\sim}{a}}}{2}$		
$\underset{\sim}{c} = \dfrac{\cancel{2}\underset{\sim}{b}(1-i) + \cancel{2}\underset{\sim}{a}i}{\cancel{2}}$		
$\underset{\sim}{c} = \underset{\sim}{b}(1-i) + \underset{\sim}{a}i$		

MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS

METHOD 2

To get $\overrightarrow{AC}$, rotate $\overrightarrow{AB}$ clockwise by $\frac{\pi}{4}$ and enlarge by a factor of $\sqrt{2}$

$$\overrightarrow{AC} = \sqrt{2}\,\overrightarrow{AB}\left(\cos\left(-\tfrac{\pi}{4}\right) + \sin\left(-\tfrac{\pi}{4}\right)\right)$$
$$= \overrightarrow{AB} \times \sqrt{2}\left(\cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{4} i\right)$$
$$\overrightarrow{AC} = \overrightarrow{AB} \times \sqrt{2}\left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}} i\right)$$
$$\underset{\sim}{c} - \underset{\sim}{a} = (\underset{\sim}{b} - \underset{\sim}{a})(1 - i)$$
$$\underset{\sim}{c} = \underset{\sim}{b} - \underset{\sim}{b}i - \underset{\sim}{a} + \underset{\sim}{a}i + \underset{\sim}{a}$$
$$\underset{\sim}{c} = \underset{\sim}{b}(1 - i) + \underset{\sim}{a}i$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}| \cos\tfrac{\pi}{4}$$
$$(\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = AB \times AC \times \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} i\right)$$
$$= AB \times \sqrt{2}\,AB\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} i\right)$$
$$\underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{a} \cdot \underset{\sim}{a} = AB^2 (1 + i)$$

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) $\displaystyle\int_2^3 \sqrt{\frac{x-2}{4-x}}\,dx$		

$x = 2 + 2\cos^2\theta$

$\dfrac{dx}{d\theta} = -4(\cos\theta)\sin\theta$

$x = 3,\quad 3 = 2 + 2\cos^2\theta$

$\cos^2\theta = \frac{1}{2}$

$\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

$x = 2,\quad 2 = 2 + 2\cos^2\theta$

$\cos^2\theta = 0$

$\theta = \frac{\pi}{2}$

OR

$x = 2 + 2 \times \frac{1}{2}(\cos 2\theta + 1)$

$x = 2 + \cos 2\theta + 1$

$x = 3 + \cos 2\theta$

$\dfrac{dx}{d\theta} = -2\sin 2\theta$

$= -2 \times 2\sin\theta\cos\theta$

$= -4\sin\theta\cos\theta$

(1) Correct bounds

$$= \int_{\pi/2}^{\pi/4} \sqrt{\frac{2\cos^2\theta}{4-2-2\cos^2\theta}} \times -4\cos\theta\sin\theta\,d\theta$$

$$= \int_{\pi/2}^{\pi/4} \sqrt{\frac{\cancel{2}\cos^2\theta}{\cancel{2}(1-\cos^2\theta)}} \times -4\cos\theta\sin\theta\,d\theta$$

$$= \int_{\pi/2}^{\pi/4} \sqrt{\frac{\cos^2\theta}{\sin^2\theta}} \times -4\cos\theta\sin\theta\,d\theta$$

(1) Correct set up of integral.

$$= \int_{\pi/2}^{\pi/4} \frac{\cos\theta}{\cancel{\sin\theta}} \times -4\cos\theta\,\cancel{\sin\theta}\,d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$$= -4\int_{\pi/2}^{\pi/4} \cos^2\theta\,d\theta$$

(1) Correct simplification of integral.

$$= \frac{4}{2}\int_{\pi/4}^{\pi/2} \cos 2\theta + 1\,d\theta$$

$$= 2\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\pi/4}^{\pi/2}$$

(1) Correctly integrates and gives correct solution.

$$= 2\left[\frac{1}{2}\sin\pi + \frac{\pi}{2} - \left(\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4}\right)\right]$$

$$= 2\left[\frac{\pi}{2} - \frac{1}{2}\times 1 - \frac{\pi}{4}\right]$$

Some half marks awarded for minor errors

$$= \pi - 1 - \frac{\pi}{2}$$

$$= \frac{\pi}{2} - 1$$

MATHEMATICS EXTENSION 2 – QUESTION 15

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
d) $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$ $u = x^n$ $v' = \cos x$ $u' = nx^{n-1}$ $v = \sin x$	1	Correct use of integration by parts applied.
$= \left[x^n \sin x\right]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$ $u = x^{n-1}$ $v' = \sin x$ $u' = (n-1)x^{n-2}$ $v = -\cos x$		
$= \left(\frac{\pi}{2}\right)^n \sin\frac{\pi}{2} - 0 - n\left[\left[-x^{n-1}\cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\cos x \, dx\right]$		
$I_{n-2} = \int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx$	1	Correct second use of integration by parts applied.
$= \left(\frac{\pi}{2}\right)^n - n\left\{\left[-\left(\frac{\pi}{2}\right)^{n-1}\cos\frac{\pi}{2} - 0\right] + (n-1)I_{n-2}\right\}$		
$= \left(\frac{\pi}{2}\right)^n - n(0-0) - n(n-1)I_{n-2}$		
$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$ $n \geq 2$	1	Correctly proves I_n.
(ii) $I_4 = \left(\frac{\pi}{2}\right)^4 - 4(4-1)I_2$		
$= \left(\frac{\pi}{2}\right)^4 - 12\left[\left(\frac{\pi}{2}\right)^2 - 2(2-1)I_0\right]$	1	Correct substitution of $n=4$ into I_n
$= \left(\frac{\pi}{2}\right)^4 - 12\left[\left(\frac{\pi}{2}\right)^2 - 2\times\int_0^{\frac{\pi}{2}} x^0 \cos x \, dx\right]$		
$= \left(\frac{\pi}{2}\right)^4 - 12\left\{\left(\frac{\pi}{2}\right)^2 - 2\times\left[\sin x\right]_0^{\frac{\pi}{2}}\right\}$		
$= \left(\frac{\pi}{2}\right)^4 - 12\left[\left(\frac{\pi}{2}\right)^2 - 2\times 1\right]$	1	Convincingly proves result, including when $n=0$.
$= \left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24$		

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) i) Aim: To prove $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$		This part was not done very well
Let $u = a - x$		
$\frac{du}{dx} = -1$		
$dx = -du$		
When $x = a$, $u = 0$		
$x = 0$, $u = a$		
Now $RHS = \int_0^a f(a-x)\,dx$		½ – correct limits of integration
$= \int_a^0 f(u)\,.\,-du$	1 mk	½ – correct u substitution
$= \int_0^a f(u)\,du$	½	
$= \int_0^a f(x)\,dx$ (since u, x are dummy variables)	½	2
$= LHS$		
Method 2		
Let $u = a - x$, then $x = a - u \Rightarrow dx = -du$ $x = 0 \Rightarrow u = a$ $x = a \Rightarrow u = 0$ $LHS = \int_0^a f(x)\,dx = \int_a^0 f(a-u)(-du)$ $= -\int_a^0 f(a-u)\,du$ $= \int_0^a f(a-u)\,du$ $= \int_0^a f(a-x)\,dx$, since u, x are dummy variables $= RHS$ as required		

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) ii)		
Let $x = \sin\theta$ $dx = \cos\theta\, d\theta$ $x = 0 \Rightarrow \theta = 0$, $x = 1 \Rightarrow \theta = \frac{\pi}{2}$ Let $I_1 = \int_0^1 \frac{dx}{x+\sqrt{1-x^2}}$, then $I_1 = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta} d\theta$	1	½ - correct limits of integration ½ - correct integrand after substitution
$= \int_0^{\pi/2} \frac{\cos(\pi/2-\theta)}{\sin(\pi/2-\theta)+\cos(\pi/2-\theta)} d\theta$, using part (i) $= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta+\sin\theta} d\theta$ $= I_2$	1	correct substitution of property in (i).
Since $I_1 + I_2 = \int_0^{\frac{\pi}{2}} \frac{\cos\theta+\sin\theta}{\sin\theta+\cos\theta} d\theta$ $= \int_0^{\frac{\pi}{2}} 1 d\theta$ $= [\theta]$ $= \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$ But $I_1 = I_2$ $2I_1 = \frac{\pi}{2}$	½	
$I_1 = \frac{\pi}{4}$ $\int_0^1 \frac{dx}{x+\sqrt{1-x^2}} = \frac{\pi}{4}$	½	3
b) $\vec{PO} = \begin{pmatrix} -8 \\ -6 \\ 0 \end{pmatrix}$ $\quad \vec{PR} = \begin{pmatrix} x-8 \\ y-6 \\ z \end{pmatrix}$		Some students wrote $\vec{PO}$ as $\begin{pmatrix} 8 \\ 6 \\ 0 \end{pmatrix}$ which brought on errors.
Using the dot product, $\vec{PO}\cdot\vec{PR} = \lvert\vec{PO}\rvert\lvert\vec{PR}\rvert\cos\frac{\pi}{3}$ ---- (*) $\vec{PO}\cdot\vec{PR} = -8(x-8) - 6(y-6) + 0\times z$ $= -8x + 64 - 6y + 36$ $= -8x - 6y + 100$	½	
and $\lvert\vec{PO}\rvert = \sqrt{(-8)^2 + (-6)^2}$ $= \sqrt{64+36}$ $= \sqrt{100}$ $= 10$ and $\lvert\vec{PR}\rvert = 10$ (given)		

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
From dot product $(*)$		
$-8x - 6y + 100 = 10 \times 10 \times \cos\frac{\pi}{3}$		
$-8x - 6y + 100 = 100 \times \frac{1}{2}$		
$-8x - 6y + 100 = 50$		
$\therefore\ 4x + 3y = 25$ ——— (1)	$\frac{1}{2}$	
Also $\overrightarrow{QO} = \begin{pmatrix} -6 \\ -8 \\ 0 \end{pmatrix}$ and $\overrightarrow{QR} = \begin{pmatrix} x-6 \\ y-8 \\ z \end{pmatrix}$		
Dot product		
$\overrightarrow{QO} \cdot \overrightarrow{QR} = \lvert\overrightarrow{QO}\rvert \lvert\overrightarrow{QR}\rvert \cos\frac{\pi}{3}$ $(**)$		
Now $\overrightarrow{QO} \cdot \overrightarrow{QR} = -6(x-6) + -8(y-6)$		
$= -6x + 36 - 8y + 64$		
$= -6x - 8y + 100$	$\frac{1}{2}$	
and $\lvert\overrightarrow{QO}\rvert = \sqrt{(-6)^2 + (-8)^2}$		
$= 10$		
and $\lvert\overrightarrow{QR}\rvert = 10$ given sub $(**)$		
$-6x - 8y + 100 = 10 \times 10 \times \frac{1}{2}$		
$-6x - 8y + 100 = 50$		
$-6x - 8y = -50$		
$3x + 4y = 25$ ...... (2)	$\frac{1}{2}$	
(1) $\times 4$, (2) $\times 3$		
$16x + 12y = 100$ ...... (3)		
$9x + 12y = 75$ ...... (4)		
(3) $-$ (4) $7x = 25$		
$x = \frac{25}{7}$ sub in (1)		
$4\left(\frac{25}{7}\right) + 3y = 25$		
$\frac{100}{7} + 3y = 25$		

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
$100 + 21y = 175$		
$21y = 75$		
$y = \frac{75}{21}$		
$y = \frac{25}{7}$		
Now $\lvert\overrightarrow{QR}\rvert = \sqrt{(x-6)^2+(y-8)^2+z^2}$		
sub $x=y=\frac{25}{7}$ and $\lvert\overrightarrow{QR}\rvert=10$		
$10 = \sqrt{\left(\frac{25}{7}-6\right)^2+\left(\frac{25}{7}-8\right)^2+z^2}$		
$10 = \sqrt{\frac{289}{49}+\frac{961}{49}+z^2}$		
$= \sqrt{\frac{1250}{49}+z^2}$		
Squaring both sides		
$100 = \frac{1250}{49} + z^2$		
$z^2 = \frac{3650}{49}$		
$z = \frac{\sqrt{3650}}{7}$ as $z>0$		
$= \frac{5\sqrt{146}}{7}$		
$\therefore$ R is $\left(\frac{25}{7}, \frac{25}{7}, \frac{5\sqrt{146}}{7}\right)$		

NOTE: Many students identified that they needed to find $\overrightarrow{PR}$ + $\overrightarrow{QR}$, they found their magnitudes and they equated them, finding that $x=y$. However students did not know what to do next. It was best for them to use the dot product to find x and y then use both the values to find z.

See Alternative solution below

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
Alternative Solution 16 b)		
$\vec{PR} = \begin{pmatrix} x-8 \\ y-6 \\ z-0 \end{pmatrix} = \begin{pmatrix} x-8 \\ y-6 \\ z \end{pmatrix}$ $\quad \vec{QR} = \begin{pmatrix} x-6 \\ y-8 \\ z \end{pmatrix}$		
but $\lvert\vec{PR}\rvert = \lvert\vec{QR}\rvert = 10$		
Now $\lvert\vec{PR}\rvert = \sqrt{(x-8)^2+(y-6)^2+z^2}$		
$10 = \sqrt{(x-8)^2+(y-6)^2+z^2}$		
$100 = (x-8)^2+(y-6)^2+z$ $\quad$ -- (1)		
Also		
$\lvert\vec{QR}\rvert = \sqrt{(x-6)^2+(y-8)^2+z^2}$		
$10 = \sqrt{(x-6)^2+(y-8)^2+z^2}$		
$100 = (x-6)^2+(y-8)^2+z^2$ $\quad$ -- (2)		
(1) = (2)		
$(x-8)^2+(y-6)^2+z^2 = (x-6)^2+(y-8)^2+z^2$		
$x^2-16x+64+y^2-12y+36 = x^2-12x+36+y^2-16y+64$	1	
$-16x-12y = -12x-16y$		
$-4x = -4y$		
$x = y$	1	
Now $\vec{QO} = \begin{pmatrix} -6 \\ -8 \\ 0 \end{pmatrix}$		

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
Using the dot product		
$\overrightarrow{QR} \cdot \overrightarrow{QO} = \lvert\overrightarrow{QR}\rvert \cdot \lvert\overrightarrow{QO}\rvert \cos\frac{\pi}{3}$		
$-6(x-6)-8(y-8) = 10\times 10\times\frac{1}{2}$	½	
$-6x+36-8y+64 = 50$		
$-6x-8y = -50$		
$6x+8y = 50$		
$3x+4y = 25$		
but $x = y$		
ie $3x+4x = 25$		
$7x = 25$		
$x = \frac{25}{7}$	½	
and $y = \frac{25}{7}$		
For z, sub x & y in (1)		
$100 = \left(\frac{25}{7}-8\right)^2 + \left(\frac{25}{7}-6\right)^2 + z^2$		
$100 = \frac{961}{49} + \frac{289}{49} + z^2$		
$z^2 = \frac{3650}{49}$		
$z = \pm\sqrt{\frac{3650}{49}}$		
but $z>0$		
$\therefore z = \sqrt{\frac{3650}{49}}$		
$= \frac{5\sqrt{146}}{7}$		
$\therefore R$ is $\left(\frac{25}{7}, \frac{25}{7}, \frac{5\sqrt{146}}{7}\right)$	1	

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
c) Given $\frac{x+y}{2} \geq \sqrt{xy}$		This part was not well done
Let $x \to a$, $y \to b$		
$\therefore \frac{a+b}{2} \geq \sqrt{ab}$		
i.e $a+b \geq 2\sqrt{ab}$ --- (1)		
Similarly $a+c \geq 2\sqrt{ac}$ --- (2)		
$b+c \geq 2\sqrt{bc}$ --- (3)		
By adding (1), (2) and (3) we get		
$a+b+a+c+b+c \geq 2\sqrt{ab}+2\sqrt{ac}+2\sqrt{bc}$		
$2a+2b+2c \geq 2(\sqrt{ab}+\sqrt{ac}+\sqrt{bc})$		
$\cancel{2}(a+b+c) \geq \cancel{2}(\sqrt{ab}+\sqrt{ac}+\sqrt{bc})$		
$a+b+c \geq \sqrt{ab}+\sqrt{ac}+\sqrt{bc}$ --- (4)	1	
Now consider the expansion:		
$(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 = a+b+c+2\sqrt{ab}+2\sqrt{ac}+\sqrt{bc}$ --- (5)	1	
Add $2\sqrt{ab}+2\sqrt{ac}+2\sqrt{bc}$ to both sides of (4) we get		
$a+b+c+2\sqrt{ab}+2\sqrt{ac}+2\sqrt{bc} \geq \sqrt{ab}+\sqrt{ac}+\sqrt{bc}+2\sqrt{ab}+2\sqrt{ac}+2\sqrt{bc}$	1	
$\therefore$ Substituting LHS of (5) we get		
$(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \geq 3\sqrt{ab}+3\sqrt{ac}+3\sqrt{bc}$		
$(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \geq 3(\sqrt{ab}+\sqrt{ac}+\sqrt{bc})$		3 marks
3 Marks: Full, clear and well structured proof		
2 Marks: – Adding all AM/GM inequalities and dividing both sides by 2. – considering the expansion. $(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 = a+b+c+2\sqrt{ab}+2\sqrt{ac}+2\sqrt{bc}$		
1 Mark: Adding all AM/GM inequalities & dividing both sides by 2.		

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
ii) From (i) $(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \geq 3(\sqrt{ab}+\sqrt{ac}+\sqrt{bc})$ --- (*)		
Let $a=m^6p^6$, $b=m^6r^6$, $c=p^6r^6$ sub in (*)		After giving the subst. for a, b and c,
$\left(\sqrt{m^6p^6}+\sqrt{m^6r^6}+\sqrt{p^6r^6}\right)^2 \geq 3\left(\sqrt{m^6p^6m^6r^6}+\sqrt{m^6p^6p^6r^6}+\sqrt{m^6p^6r^6r^6}\right)$	1	many students did not include the LHS of this line in their proof and went straight to the third line.
$(m^3p^3+m^3r^3+p^3r^3)^2 \geq 3\left(\sqrt{m^{12}p^6r^6}+\sqrt{m^6p^{12}r^6}+\sqrt{m^6p^6r^{12}}\right)$		
$(m^3p^3+m^3r^3+p^3r^3)^2 \geq 3(m^6p^3r^3+m^3p^6r^3+m^3p^3r^6)$	1	
$\geq m^6p^3r^3+m^3p^6r^3+m^3p^3r^6$		
$(m^3p^3+m^3r^3+p^3r^3)^2 \geq m^3p^3r^3(m^3+p^3+r^3)$		2
NOTE: This is a proof question and requires you to show <u>ALL</u> steps. Please attempt more proof/show type questions and practise showing all steps in your solutions.		